In order to clarify the situation with regard to the validity of the P-L-K technique for problems of the type discussed by Jischke it is worthwhile to examine these solutions in more detail. This is conveniently done with his example problem. First consider the case of Γ small and, for convenience, $\alpha = 0$. The P-L-K solution to third order in Γ is $y(x,\Gamma) = 1 + \Gamma \ln\xi$ where $x = \xi - \Gamma\xi(\ln\xi)^2 + \Gamma^2\xi(1 + \frac{1}{2}\ln\xi)(\ln\xi)$.[3] This particular form was obtained by arbitrarily setting the coefficients of Γ^2 and Γ^3 in the expansion of $y(x;\Gamma)$ to zero. A comparison of the second-order P-L-K solution (neglects the Γ^2 term in the expression for x), the third-order P-L-K solution, and the exact solution $[y = (1 - \Gamma \ln x)^{-1}]$ in the vicinity of $x = 0$ is presented in Fig. 1. The result for $\Gamma = 0.1$ is shown in part a. It is apparent that the agreement between the approximate solutions and the exact solution is quite good considering the magnitude of Γ. It is also shown that the inclusion of higher-order terms improves the accuracy. The same comparison but for $\Gamma = 0.01$ is shown in part b. For this case the third-order result is so close to the exact solution that the differences are not discernable. It is clearly illustrated that a reduction in Γ greatly improves the accuracy of the P-L-K solution. It appears then that the P-L-K solution is entirely satisfactory with the single exception that it does not provide the correct value of y at $x = 0$. Of course, that value can be obtained merely be evaluating the original differential equation at $x = 0$.

The P-L-K solution is somewhat arbitrary as the coefficients in the expansion of the independent variable are obtained by specifying that higher-order coefficients in the expansion of the dependent variable be no more singular than the lowest order singular coefficient. For this example the lowest order singular coefficient was $\ln x$. The nature of the P-L-K solution was investigated for higher order terms of the form Constant $(\ln x)$. No significant differences were obtained by allowing the arbitrary constants to assume a range of values (the constant was set to zero for the examples shown in Fig. 1).

The P-L-K technique was also applied to the example problem for α small. The manipulations were simplified by introducing the transformation $u(x) = y^{-1}(x)$. The exact solution, which is

$$u(x) = \frac{1}{\alpha}\left[\frac{1 - [(1-\alpha)/(1+\alpha)]x^{2\alpha\Gamma}}{1 + [(1-\alpha)/(1+\alpha)]x^{2\alpha\Gamma}}\right]$$

and the P-L-K solution, which is $u(x) = 1 - \Gamma \ln\xi, x = \xi + \alpha\xi[\ln\xi + \Gamma(\ln\xi)^2 + \Gamma(\ln\xi)^3/3]$ are compared in Fig. 2, for $\Gamma = 1$ and $\alpha = 0.1$. As for the case of Γ small the P-L-K technique fails to provide the correct value of u at $x = 0$, but is otherwise satisfactory. When the P-L-K technique is applied to the radiating flow problem [Jischke's Eq. (7)] one obtains the result

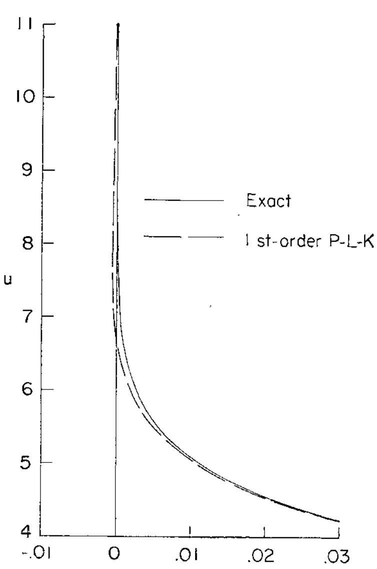

Fig. 2 Comparison of P-L-K and exact solutions for α small.

$$h_{P-L-K}(x) = h_0(\xi),\ x = \xi - \tau_s h_1(\xi)/h_0'(\xi)$$

where h_0 and h_1 are given by Jischke's Eq. (17). A similar result was presented earlier by the present author.[3]

In conclusion then it is clear that the P-L-K technique provides an entirely satisfactory solution to the present problem despite Jischke's objection that it fails to properly resolve the nature of the singularity. Moreover it has been pointed out (see, i.e., Van Dyke[5]) that the P-L-K technique when it applies is often much simpler than the MAE technique. This was demonstrated for the nongray radiating, inviscid, stagnation-region flow problem by the present author[4] who derived the "boundary-layer" form of the equations for the coupled system of energy and tangential momentum equations. The equations remained coupled when the method of MAE was applied but were uncoupled upon application of the P-L-K technique.

References

[1] Jischke, M., "Asymptotic Description of Radiating Flow Near Stagnation Point," *AIAA Journal*, Vol. 8, No. 1, Jan. 1970, pp. 96–101.

[2] Olstad, W., "Stagnation-Point Solutions for an Inviscid, Radiating Shock Layer," *1965 Heat Transfer and Fluid Mechanics Institute*, Stanford Univ. Press, 1965, p. 138.

[3] Olstad, W., "Stagnation-Point Solutions for Inviscid, Radiating Shock Layers," Ph.D. thesis, Harvard Univ., April 1966.

[4] Olstad, W., "Blunt-Body Stagnation Region Flow with Nongray Radiation Heat Transfer—A Singular Perturbation Solution," TR R-295, Nov. 1968, NASA.

[5] Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964, Chap. 6.

Reply by Author to W. B. Olstad

MARTIN C. JISCHKE*

University of Oklahoma, Norman, Okla.

OLSTAD has raised a number of questions concerning Ref. 1. We shall attempt to respond to each of these questions.

The second sentence in the second paragraph implies the determination of the correct value of the (inviscid) enthalpy of the gas adjacent to the wall is "of little practical interest." The importance of determining the correct value lies in the fact that it provides the correct outer boundary condition for the boundary-layer flow as well as the scaling for the enthalpy in the boundary layer. Indeed, only by using the correct value of the (inviscid) enthalpy of the gas at the wall does one recover the correct inverse-square-root scaling of boundary-layer thickness with Reynolds number. In passing, we note that Levey[2] has shown that the P-L-K method cannot be applied to the viscous boundary-layer problem.

Continuing in the same paragraph, it is stated that "global quantities obtained with the regular perturbation solution are not sufficiently accurate for engineering purposes." It is explicitly demonstrated in Ref. 1 that the error obtained in using the regular perturbation solution to estimate global quantities is exponentially small $[\mathcal{O}(e^{-1/\Gamma}/\Gamma)]$. This should be sufficiently accurate for engineering purposes. Also, unless an exact solution is available, it is not evident that the "improvement" obtained using the P-L-K technique is indeed an improvement. The model equation discussed in Ref. 1

Received May 4, 1970; revision received May 18, 1970.

* Assistant Professor, School of Aerospace and Mechanical Engineering. Member AIAA.

is an example where the improvement given by the P-L-K technique is illusory. In this connection, Olstad's Fig. 2 provides an excellent illustration of this point. Note that the second-order P-L-K solution for $\Gamma = 0.1$ gives a value of approximately 0.69 for $y(0)$ as compared to the exact value of zero. Similarly for $\Gamma = 0.01$, $y(0)$ equals approximately 0.9 and the convergence difficulty can be seen. That is, as we take Γ smaller and smaller, the P-L-K solution is less and less accurate. Also, the third-order P-L-K solution yields no value for $y(0)$ and gives multiple values of y for x small. This convergence difficulty persists in higher-order calculations. Furthermore, the analytical results obtained by the P-L-K solution, apart from numerical inaccuracies, has the wrong functional dependence upon the small parameter Γ. The P-L-K solution simply does not give the correct dependence upon Γ as Γ approaches zero.

Regarding the suggested relaxation of the criterion for uniform validity, it is not conventional. Also, it is difficult to interpret a uniform validity condition where one compares the exact solution at a point with the approximate solution at a *different* point. Furthermore, unless one can explicitly evaluate $\lambda(\epsilon)$—a difficult task in general—this criterion is of limited utility.

Finally, we disagree with the conclusion that the P-L-K technique provides an entirely satisfactory solution. A satisfactory solution is here taken to mean a solution which approximates the exact solution uniformly in the region of interest. The singular behavior occurs near $X = 0$ and the P-L-K technique does not correct this difficulty and does not yield a uniformly valid solution according to conventional definitions of uniform validity.[3,4] The simplicity of the P-L-K technique, while perhaps a virtue, does not insure correctness. There are several examples in the literature (see Ref. 3) where the P-L-K method gives what appears to be "reasonable" results that are, in fact, completely erroneous. It is for this reason that Lighthill[5] suggests restriction of application to hyperbolic partial differential equations.

References

[1] Jischke, M. C., "Asymptotic Description of Radiating Flow Near Stagnation Point," *AIAA Journal*, Vol. 8, No. 1, Jan. 1970, pp. 96–101.

[2] Levey, H., "The Thickness of Cylindrical Shocks and the P-L-K Method," *Quarterly of Applied Mathematics*, Vol. 17, 1959, pp. 77–93.

[3] Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964.

[4] Cole, J., *Perturbation Methods in Applied Mechanics*, Gin Blaisdell, Waltham, Mass., 1968.

[5] Lighthill, M., "A Technique for Rendering Approximate Solutions to Physical Problems Uniformly Valid," *Zeitschrift ufer Flugwiss*, Vol. 9, 1961, pp. 267–275.

Errata: "Optimized Acceleration of Convergence of an Implicit Numerical Solution of the Time-Dependent Navier-Stokes Equations"

JOE F. THOMPSON

Mississippi State University, State College, Miss.

[AIAA J. **7**, 2186–2198 (1969)]

THE word "much" should be deleted from the first sentence of the last paragraph on p. 2187. The analysis is valid for any cell Reynolds number greater than 2. The sign of the 4 inside the bracket in the numerator of the expression for $a_{p,q}$ and Eq. (14) should be negative. The sign of the 1 in the numerator of Eq. (15) and the abscissa of Fig. 1 should be negative. The points in Fig. 1 are not affected.

Received April 16, 1970.